Wydział Elektrotechniki, Automatyki i Informatyki

Autoreferat pracy doktorskiej:

Pole-free perfect control for LTI MIMO systems described in the discrete-time state-space framework

Autor:
mgr inż. Marek Krok

Promotor:
dr hab. inż. Wojciech P. Hunek

Contents

Chapter 1

Introduction

The design of multivariable control systems has gathered considerable research interest over the years. In modern technology, the ability to simultaneously obtain the desired behavior of multiple variables is a key task with a plethora of possible applications [1–6]. The most obvious example of multivariable control is an issue concerning the control of robotic manipulators, especially the position control of its working tip. Naturally, such advanced control scenarios would not be possible without proper mathematical descriptions of considered processes. Therefore, at the foundations of control theory, there are frameworks regarding system descriptions. While the polynomial-based input-output description became very complex for multivariable systems, a solution to this is a matrix-based state-space description [7–9].

With the use of the state-space framework, effective control algorithms for multivariable plants have been significantly simplified. With the admission of burdensome polynomial equations the computational effort was limited and calculation errors were avoided. Naturally, well known matrix properties can be used here to describe control systems. For example, plant stability is clearly connected with system matrix eigenvalues. Moreover, the considered system notion provides, in comparison with input-output instances, an additional state vector which reflects the inner states of plant. Additionally, the state-space description allows to assume non-zero initial conditions, which is a very useful feature. With the knowledge of both inner and outer states of system there is a wide possibility to create different control structures.

The provided system description allowed to obtain new control strategies, previously not available for systems described by transfer functions. The concept of state-feedback system has gathered wide scientific effort. With the negative gain fed from a state vector, the closed loop plant can arbitrarily be influenced by proper control design [10–12]. From simple stabilizing scenarios to advanced pole-placement instances, the state-feedback method is subject to numerous studies. The pole-placement method allows to move the eigenvalues of closed-loop system matrix over the complex plane in order to obtain desired system properties. The state of the art is to place all system poles exactly at zero, which provides extreme behavior of control instances. A recently described deadbeat response can be characterized by high control speed and accuracy. Naturally, the concept of placing all closed-loop system poles at zero have found numerous industrial applications alongside theoretic considerations [13–17]. Alongside classic deadbeat control, its alternative form focusing on the output signals was developed. The so-called output deadbeat control aims to provide output stabilization while accounting for certain limitations. Both of state-feedback control schemes mentioned here can be solved in different ways obtaining different properties under the same eigenvalues. However, the focus on obtaining desired output signals with possibly low control error seems to be an interesting idea.

The perfect control is another strategy consisting of the feedback taken from the state vector. However, in contrast to previously presented control laws, the perfect control focuses only on the control error, obtaining its possible lowest value as the output vector reaches its reference values right after the delay derived from the system description. The perfect control can be obtained by proper inverse-based state-feedback which guarantees desired system response. It has been shown that the perfect control can be further enhanced with the application of various matrix inverses [8, 18, 19]. When in cases covering square systems there is only one available solution, in the nonsquare case there is a wide range of possibilities. The use of nonunique inverses may result in different properties obtained within the minimum variance/perfect control framework [8, 19–22]. For example, the well known problem of unstable Inverse Model Control (IMC) in case of non minimum-phase systems can be overthrown with proper choice of inverses in a simple way. Moreover, thanks to nonunique inverses, the perfect control systems can gain its traits as speed or robustness. The poles of closed-loop perfect control can be placed almost arbitrarily just by application of other than minimum-norm inverses.

During the pursuit of robust, maximum-speed or/and minimum-energy perfect control, the different inverses were examined in the context of obtained closed-loop properties. Having a wide range of available tools, the idea of placing all closed-loop perfect control system poles at zero has recently been revealed. With proper selection of inverses, degrees of freedom joining properties of deadbeat and perfect control were obtained. With the closed-loop system matrices being nilpotent, the steady state occurs in almost no time. Simultaneously, with respect to the perfect control design, the output remains on the reference value with the lowest possible control error. Therefore, the pole-free perfect control became a subject of intense research efforts [23–26].

A number of analytic requirements of pole-free perfect control was presented. These requirements were divided in two categories. First category is a group of limitations and rules that need to be fulfilled before the proper control design process. A structure of control plant and its properties, such as controllability and observability were the main requirements there. On the other hand, formulas that need to be fulfilled during the control process are also discussed. More procedures enabling to calculate pole-freeing degree of freedom β of right σ-inverse were obtained by complex inverse matrix calculation. These procedures allowed to perform numerous simulation examples aiming to reveal the pole-free perfect control properties.

With efficient simulation tools, a plethora of intriguing behaviors was observed in the pole-free perfect control instances. Exactly as expected, the developed control strategy results in highly varying signals with high amplitudes, but the steady states are obtained in possibly lowest control times. However, among expected properties previously unseen energy-based irregularities have occurred. The pole-free perfect control eradicates the usual compromise between control energy and speed [27]. A single example in which the pole-free instance resulted in control energy lower than in minimum-norm approach was followed by a complex study regarding energy-based performance indices. With the use of the mentioned indices, the proper energy-based comparison was enable. Therefore, a possibility to anticipate energy outcome of different control instances was obtained with a simple analytic indices [28, 29].

As the pole-free perfect control seems to be an interesting control strategy in terms of both control speed and energy, this thesis constitutes a survey of this topic. Therefore, this work is organized as follows. At the beginning a wide mathematical background is given. The basic formulas considering system description, matrix inverses and matrices itself are presented. In Chapter 3, the topic of control strategies based on the state-feedback is reflected upon. A brief comparison of various control laws has shown that different closed-loop properties can be obtained with various state feedbacks. Then the pole-free perfect control is described in Chapter 4. A survey of pole-free requirements, traits and simulation examples presented within this chapter constitutes the main part of this thesis. Moreover, a numerical procedure allowing to obtain the pole-free perfect control under certain limitation is given in this chapter. Following the fundamental pole-free consideration, a energy-based study is presented in Chapter 5. Useful tools to evaluate pole-free and stable-pole instances in the context of state and control energies are also provided here. Final conclusions are given in the last chapter of this Ph.D. thesis.

Chapter 2

Mathematical preliminaries

In this chapter, the basic mathematical issues and definitions are presented. Although this is just a survey of the well-known rules and regularities, the content can help to understand the matter investigated throughout this thesis. A wide range of matrix properties with some theorems are discussed here. Together with basics of matrix calculus, the problem of matrix inverse is given alongside the system description, which consolidates the matrix nature of conducted study.

2.1 System representation

The system representation is crucial during design of all advanced control strategies. The framework of description considering a part of real plant is almost always simplified to a degree in regards to a real process. In this particular study the classical state-space framework originally introduced by Rosenbrock in early 70's is used [3]. This concept is especially useful for multivariable or multiple-input/multiple-output (MIMO) systems, due to its parameter matrix-based nature [30]. The disposal of polynomial matrices from the input-output descriptions led to a significant restriction of needed computional effort and unwanted calculation errors [31]. Thus, in this study the multivariable state-space framework is used to describe the linear (or rather linearized) plants.

Let the n-th order linear time-invariant (LTI) MIMO system with n_u-input variables of $\mathbf{u}(k)$ be described in the following manner

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k), \quad \mathbf{x}(0) = \mathbf{x}_0, \tag{2.1}$$

where $\mathbf{A} \in \Re^{n \times n}$ and $\mathbf{B} \in \Re^{n \times n_u}$ denote the state and input matrices, respectively, while $\mathbf{x}(k)$ covers the state vector in discrete time k. This equation is used to describe the auto-regressive part of the plant. Within this formula the whole dynamics is described. In several cases, this equation is treated as a standalone system description focusing on the internal states of the control plant. However, to fully describe the system, together with the outputs, a second expression is needed. Hereafter, in this consideration the n_y-output vector is in the form of

$$\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k), \tag{2.2}$$

where output matrix $\mathbf{C} \in \Re^{n_y \times n}$ is assumed. The parameters of aforementioned matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are obtained during the process of identification [32–34]. Of course, the model described in the discrete-time state-space framework can employ different properties covering the behaviors of the considered plant. One of most frequently discussed trait of the system is its stability.

Remark 2.1 *In plethora of literature the feed-forward matrix* $\mathbf{D}$ *is considered. In this case the output equation (2.2) is expressed as follows*

$$\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) + \mathbf{D}\mathbf{u}(k),$$

with $\mathbf{D} \in \Re^{n_y \times n_u}$*. Nevertheless, in this study this matrix is omitted due to targeting a clear notion of more complex equations presented later in the thesis.*

For years many different definitions of stability have been proposed for a wide range of considered systems [35, 36]. Nevertheless, in this study the stability of open-loop system is determined using the characteristic polynomial in the form of

$$p_{\mathbf{A}}(\lambda) = det(\lambda \mathbf{I}_{\mathrm{n}} - \mathbf{A}), \tag{2.3}$$

where $\mathbf{I}_{\mathrm{n}}$ denotes n-by-n identity matrix, and λ is some complex variable. As the matter presented in this thesis regards time-invariant plants with constant single unit delay, this framework is sufficient for the research to be presented. Moreover, the roots of characteristic polynomial, often called poles of the open-loop system, can be calculated according to the following characteristic equation

$$det(\lambda \mathbf{I}_{\mathrm{n}} - \mathbf{A}) = 0. \tag{2.4}$$

Another important plant behaviors are observability and controllability. The aforementioned properties are often crucial during the design of control schemes. For example, the plant is needed to be controllable during many control problems, such as optimal control problem or stabilization of unstable systems by proper feedback [37–39]. Notwithstanding, in some cases the controlability is not necessary. Sometimes there is consent for plant having uncontrollable modes, but therefore these modes need to be stable.

2.2 Matrix Inverses

The calculation of proper matrix inverse is a highly developed research area which has atttracted considerable interest during past years. As an issue that requires considerable computational effort and mathematical background, this topic has combined the developments coming from both theoretical and practical approaches [8, 9, 21, 40, 41]. Beginning at the inverse calculation considering the simple well-posed low rank matrices, this issue evolved to solve tasks of unimaginable sizes [18, 41]. Nevertheless, there are still a plethora of scientific projects aiming the improvement in matrix inverse calculations and its possible applications [42]. Moreover, numerous advancements that have originated from studies considering square matrices can be adapted to the nonsquare cases.

The basics of matrix inverse problem have been defined as a quest to find such matrix $\mathbf{M}^{-1}$ for given matrix $\mathbf{M}$ that its product will be equal to a unity or identity matrix $\mathbf{I}_{\mathrm{n}}$. The calculation of square inverse matrices is a complex problem, relying on the numeric procedures devoted to this area. Especially, the issue intensifies with matrices that are not well-posed, triangular matrices, sparse or in general, matrices that have inverses which are not well-conditioned.

However, the admission of nonsquare matrices for calculation of their inverses, together with their square analogues, has found wide implementation in control theory, thus this subsection is devoted to them. Moreover, in the effort considering nonsquare matrices, square inverses are often used with their application to some Hermitian forms, therefore the mentioned area is not to be omitted in this consideration [43].

Basing on the properties of square inverses, the more complex nonsquare instances can be indicated. The general case of pseudo- or generalized-inverse $\mathbf{M}^{+}$ needs to satisfy four Moore-Penrose requirements. The first two conditions are as follows

$$\mathbf{M}\mathbf{M}^{+}\mathbf{M} = \mathbf{M}, \tag{2.5}$$

and

$$\mathbf{M}^{+}\mathbf{M}\mathbf{M}^{+} = \mathbf{M}^{+}, \tag{2.6}$$

where $\mathbf{M}\mathbf{M}^{+} = \mathbf{I}_{\mathrm{n}}$ is not necessary to hold, in general. The last two requirements are

$$(\mathbf{M}\mathbf{M}^{+})^{\mathrm{T}} = \mathbf{M}\mathbf{M}^{+}, \tag{2.7}$$

and

$$(\mathbf{M}^{+}\mathbf{M})^{\mathrm{T}} = \mathbf{M}^{+}\mathbf{M}. \tag{2.8}$$

With fulfillment of all the above relations, the generalized inverse can be established. Of course, different frameworks considering generalized inverse have been known for years.

In the case of full rank parameter matrices, some simpler rules can be preserved. These are dealing separately with the so-called right and left inverse of nonsquare matrix. The right inverse is needed to fulfill the following relation

$$\mathbf{M}\mathbf{M}^{\mathrm{R}} = \mathbf{I}_{\mathrm{n}}, \tag{2.9}$$

whereas the left inverse submits to the equation in form of

$$\mathbf{M}^{\mathrm{L}}\mathbf{M} = \mathbf{I}_{\mathrm{n}}. \tag{2.10}$$

These inverses can be calculated in different manners. The main distinctions between various inverse frameworks consider such properties as rounding and cut-off errors, computational effort or complexity. Next, some of the most commonly used inverses will be presented in order to clarify their notions and features.

2.2.1 Unique T-inverse

A first of considered inverses is the unique T-inverse, which has only one solution for specified nonsquare matrix. The uniqueness following this inverse entails that the T-inverse is often used as a benchmark for other inverses [8]. Consider a $\mathbf{M} \in \Re^{m\times n}$ arbitrary matrix. The right T-inverse of $\mathbf{M}$ can be calculated according to the following equation

$$\mathbf{M}_0^{\mathrm{R}} = \mathbf{M}^{\mathrm{T}}(\mathbf{M}\mathbf{M}^{\mathrm{T}})^{-1}, \tag{2.11}$$

while the left T-inverse can be obtained from

$$\mathbf{M}_0^{\mathrm{L}} = (\mathbf{M}^{\mathrm{T}}\mathbf{M})^{-1}\mathbf{M}^{\mathrm{T}}. \tag{2.12}$$

In various tasks the T-inverse, thanks to its minimum-norm property, results in least squares or minimum-norm solutions. Nevertheless, it has been shown that in some distinguished manners this inverse can be outperformed by the nonunique ones, which will be presented in the upcoming subsections.

2.2.2 Nonunique σ-inverse

In this subsection the σ-inverse is shortly discussed. The mentioned nonunique inverse gathered considerable research interest due to its clear basics and relatively low computational effort. The application in control theory has proven the σ-inverse to have desired properties in terms of stabilization, robustification and energy optimization [44, 45]. Its right-invertible equivalent sounds as follows

$$\mathbf{M}^{\mathrm{R}} = \beta^{\mathrm{T}}(\mathbf{M}\beta^{T})^{-1}, \tag{2.13}$$

where $\beta \in \Re^{m\times n}$ is the so-called degree of freedom. This degree of freedom allows to form the obtained inverse to obtain the desired properties with almost no limitation. The only requirement is for the matrix product $(\mathbf{M}\beta^{\mathrm{T}})$ to be right-invertible. Of course, for the matrices with $m < n$ the valid is left σ-inverse in the following form

$$\mathbf{M}^{\mathrm{L}} = (\beta^{\mathrm{T}}\mathbf{M})^{-1}\beta^{\mathrm{T}}, \tag{2.14}$$

with ($\beta^{\mathrm{T}}\mathbf{M}$) being a full rank. Below an exemplary right σ-inverse is shown. It is worth emphasizing that the nonunique inverse can result in infinite number of different numeric instances of matrices, all fulfilling the requirements imposed to the inverse matrices. Thus the 'good' or 'bad' degrees of freedom provide of the specific application of inverse to be obtained.

2.2.3 Nonunique H-inverse

Another of the inverses used in this thesis is the nonunique H-inverse, which is far more complex than previous ones [46]. In this case, at the beginning, the Singular Value Decomposition (SVD) has to be calculated in the following form

$$\mathbf{M} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^{\mathrm{T}}, \tag{2.15}$$

where $\mathbf{\Sigma}$ contains eigenvalues, $\mathbf{U}$ and $\mathbf{V}$ are right- and left- eigenvector, respectively. Basing on the well-known rule of inverse calculation for square matrix product the wanted unique inverse can be obtained as follows

$$\mathbf{M}^{-1} = (\mathbf{V}^{\mathrm{T}})^{-1}\mathbf{\Sigma}^{-1}\mathbf{U}^{-1}. \tag{2.16}$$

Nevertheless, in the nonsquare case, the degrees of freedom in form of $\mathbf{L}$ reveal in the inverse procedure. Having a matrix with $m \neq n$ we calculate the SVD-related Σ matrix in the form of

$$\mathbf{\Sigma} = \begin{bmatrix} \mathbf{\Lambda} & \mathbf{0} \end{bmatrix} \text{ or } \mathbf{\Sigma} = \begin{bmatrix} \mathbf{\Lambda} \\ \mathbf{0} \end{bmatrix}, \tag{2.17}$$

with $\mathbf{0}$-matrices of dimensions depending whether the matrix $\mathbf{M}$ is right- or left-invertible. Of course, in the square $\mathbf{M}$ case the presented zero matrix does not appear. Moreover, $\mathbf{\Lambda}$ is a diagonal matrix involving the eigenvalues of $\mathbf{M}$ in the following manner

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & \lambda_n \end{bmatrix}. \tag{2.18}$$

Thus, during the pursuit of nonsquare matrix inverse based on the SVD procedure, the challenge is to find such $\mathbf{\Sigma}$ inverse that results in a required numerical solution. Interestingly, since the inverse of matrix $\mathbf{\Lambda}$ is rather a trivial problem, the inverse of nonsquare $\mathbf{\Sigma}$ can also be obtained in a simple way according to

$$\mathbf{\Sigma}^{\mathrm{R}} = \begin{bmatrix} \mathbf{\Lambda}^{-1} \\ \mathbf{L} \end{bmatrix} \text{ or } \mathbf{\Sigma}^{\mathrm{L}} = \begin{bmatrix} \mathbf{\Lambda}^{-1} & \mathbf{L} \end{bmatrix}, \tag{2.19}$$

where $\mathbf{L}$ is the so-called degree of freedom, depending on the sizes of original matrices $\mathbf{M}$. Here the degrees of freedom reveal crucial feature in the nonunique approach to nonsquare matrix inverses. This ability to manage the solution of inverse problem is usually used to overcome the numerical issues. Of course, in the case of H-inverse such inconveniences are limited to the inverse of square matrix λ, so there is no way to improve them in this context. However, the whole H-inverse itself seems to be robust in terms of limitations tied to degrees of freedom. In contrast to the β of σ-inverse, the matrix $\mathbf{L}$ can be chosen arbitrarily, without any restrictions. Back to the inverse, in case of right-invertible matrices, the right H-inverse can be obtained with the following formula

$$\mathbf{M}^{\mathrm{R}} = (\mathbf{V}^{\mathrm{T}})^{-1}\mathbf{\Sigma}^{\mathrm{R}}\mathbf{U}^{-1}, \tag{2.20}$$

whilst the left H-inverse is calculated in the following way

$$\mathbf{M}^{\mathrm{L}} = (\mathbf{V}^{\mathrm{T}})^{-1}\mathbf{\Sigma}^{\mathrm{L}}\mathbf{U}^{-1}. \tag{2.21}$$

Of course, the usage of H-inverse needs to be suited to a particular application. This can be done by choosing a proper degree of freedom, satisfying more then just computational requirements imposed on the obtained solution. It is worth mentioning, that this particular inverse has recently found innovative application in the multidimensional wireless data transmission.

2.3 Nilpotent matrices

During the stability analysis of control plants the ratio of energy stored inside the system is crucial. If the ratio is above unity, the system drifts towards its unstable regions. In the opposite scenario, when this coefficient is lower then unity, the internal plant energy will converge to zero, assuming that no additional power is fed to the system. Taking into consideration the autoregressive systems, it is clear now that for a high enough sampling time k, all signals will be equal to zero. Same as in linear algebra, there is a concept of matrices that for high enough power the matrix converges to a zero matrix. The definition of property called nilpotency can be found in open literature. In the largest part of literature the nilpotent matrix is understood in terms of matrix having its power k, for which the following expression holds

$$\mathbf{M}^k = \mathbf{0}. \tag{2.22}$$

It is worth mentioning that the integer coefficient k is often called an index of nilpotence. Moreover, the nilpotency of matrix $\mathbf{M} \in \Re^{\mathrm{n}\times\mathrm{n}}$ is associated with a number of properties. The characteristic polynomial of nilpotent matrix is in form of

$$p_{\mathbf{M}}(\lambda) = \lambda^n. \tag{2.23}$$

From the above the next property can be obtained. Having such a characteristic polynomial, it is clear that all of matrix eigenvalues are equal to zero. Next property states that the trace of any matrix power k submits to the following equation

$$tr(\mathbf{M}^k) = 0. \tag{2.24}$$

However, these properties are not obtained for all matrices with $\mathbf{M}^k = 0$. It is rather obvious, that matrices with eigenvalues within unitary disc have such k for which the matrix power converges to zero matrix. In such case, the matrix can be called nilpotent, but the nilpotence index can be very high. These circumstances leading to merge of convergent and nilpotent matrices.

Therefore, in this thesis, the matrix is treated as nilpotent iff for square matrix $\mathbf{M} \in \Re^{\mathrm{n}\times\mathrm{n}}$ there exists an integer power $k \leq n$ that satisfies Eq. (2.22). Such approach restrict the usage of the term 'nilpotent' beyond just convergent matrices. Moreover, this limitation also provides the properties mentioned earlier in this subsection.

2.4 Cayley-Hamilton theorem

In linear algebra, the Cayley-Hamilton theorem deals with square matrices over a commutative ring. Therefore, the particular parameter matrices can be considered. The mentioned theorem states, that examined matrices fulfill their own characteristic equations. With the characteristic polynomials defined as in Eq. (2.3) the Cayley-Hamilton theorem imposes the following relation

$$p_{\mathbf{M}}(\mathbf{M}) = 0. \tag{2.25}$$

Naturally, the Cayley-Hamilton theorem can easily be demonstrated. Having the characteristic polynomial of complex variable λ in the form of $p_{\mathbf{M}}(\lambda) = det(\lambda\mathbf{I}_{\mathrm{n}} - \mathbf{M})$, let us make a substitution $\lambda = \mathbf{M}$. The result is immediately obtained in the form of $det(\mathbf{M}\mathbf{I}_{\mathrm{n}} - \mathbf{M}) = \mathbf{0}$.

It is worth mentioning that the Cayley-Hamilton theorem also covers the polynomial and rational matrices or even fractional-order systems. More information considering this matter can be found, for example, in Ref. [47].

The main advantage coming from the Cayley-Hamilton theorem is a fact, that the minimal polynomial of given matrix is a divider of its characteristic polynomial. This feature can be used, for example, in calculating the matrix Jordan normal form described in the next section.

2.5 Jordan normal form

Matrix diagonalization is a well-known process widely used in the mathematical transformations, proofs and other calculations. In the simplest manner, the diagonal form can be obtained with the following relation

$$diag(\mathbf{A}) = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}. \tag{2.26}$$

Of course, the diagonal matrix $diag(\mathbf{A})$ contains all eigenvalues of matrix $\mathbf{A}$ on the main diagonal, with all other entries equal to 0. In the control theory, the diagonal form is often applied to the system matrix $\mathbf{A}$ (see Eq. (2.1)) in order to reveal system eigenvalues.

Nevertheless not all matrices are diagonizable, thus this consideration is not applicable in every cases. Although the requirements that the matrix needs to fulfill will not be presented here, the remedy in form of generalized diagonalization is briefed below. Two different strategies can be considered under the term generalized diagonalization. The first one is well-known SVD procedure, mentioned in Eq. (2.15), in which the matrix with original eigenvalues is obtained. The second one is a Jordan normal form, which is used in this thesis along with other matrix transformations. During the analysis of matrices with multiple eigenvalues, multiple zero eigenvalues, non-full rank matrices, or singular matrices, the strict diagonal form often does not exist. In such cases, the almost diagonal Jordan normal form can be used.

The Jordan normal form, often called Jordan canonical form, can be obtained with the use of the following equation

$$J(\mathbf{A}) = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}, \tag{2.27}$$

for a certain proper matrix $\mathbf{P}$. The mentioned, almost diagonal form is of the following structure

$$J(\mathbf{A}) = \begin{bmatrix} \mathbf{J_1} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{J_2} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{J_n} \end{bmatrix}, \tag{2.28}$$

where $\mathbf{0}$ denotes a zero block matrix. Additionally, $\mathbf{J_{i=1,\dots,n}}$ stands for associated with i-th eigenvalue Jordan matrix in the form of

$$\mathbf{J_i} = \begin{bmatrix} \lambda_i & 1 & \dots & 0 \\ 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & \lambda_i \end{bmatrix}. \tag{2.29}$$

2.6 Sum of matrix series

Similarly to the paradigm of the classical mathematical meaning, if matrices are a part of geometric series, their sums can be calculated as long as the series are convergent. In the case of matrix geometric series, the convergence is related to the eigenvalues of the respective matrices. If all eigenvalues are contained within the unit disc, then the series converge to a zero matrix for a high enough matrix power. In such a case we have

$$\sum_{n=0}^{\infty} \mathbf{M}^n = \mathbf{S_M}. \tag{2.30}$$

The sum can be obtained with some mathematical operations. After defining the expression

$$\mathbf{S_M} = \mathbf{I}_\mathrm{n} + \mathbf{M} + \mathbf{M}^2 + \mathbf{M}^3 + \ldots \tag{2.31}$$

multiplicated by $\mathbf{M}$ the following formula holds

$$\mathbf{MS_M} = \mathbf{M} + \mathbf{M}^2 + \mathbf{M}^3 + \ldots. \tag{2.32}$$

Thus, we easily obtain

$$(\mathbf{I}_\mathrm{n} - \mathbf{M})\mathbf{S_M} = \mathbf{I}_\mathrm{n}, \tag{2.33}$$

and finally

$$\mathbf{S_M} = (\mathbf{I}_\mathrm{n} - \mathbf{M})^{-1}, \tag{2.34}$$

recalled with factor $(\mathbf{I}_\mathrm{n} - \mathbf{M})$ being a full rank matrix. The application of mentioned rule can be found in brief numerical example presented below.

2.7 Matrix exponential

The matrix exponential is a natural expansion of the classical exponential function into the matrix calculus. While the classical exponential can be calculated as a sum of power series as follows

$$e^{\mathbf{x}} = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \tag{2.35}$$

the same can be applied to the matrix exponential. Thus, it can be obtained using the following formula

$$e^{\mathbf{M}} = \sum_{k=0}^{\infty} \frac{1}{k!}\mathbf{M}^k. \tag{2.36}$$

In the nilpotent case, the matrix exponential can be obtained directly from the series expansion

$$e^{\mathbf{M}} = \mathbf{M}^0 + \mathbf{M}^1 + \ldots + \frac{1}{k-1}\mathbf{M}^{k-1}, \tag{2.37}$$

where k denotes the rank of nilpotence as in Eq. (2.22). Additionally, there is yet one interesting equation considering the matrix exponential. The Jacobi formula represents a connection between determinant of matrix exponential and exponential of matrix trace as follows

$$det(e^{\mathbf{M}}) = e^{tr(\mathbf{M})}. \tag{2.38}$$

Chapter 3

State-feedback control strategies

During the design of control schemes, the main goal is to shape the plant response and characteristics to a desired state. In a great number of applications, this can be obtained by means of proper feedback. This feedback can be taken from both state and output variables. The output feedback is one of the main tools during the design of controllers for transfer functions in the input-output system descriptions. On the other hand, the state-space framework provides additional state vector. Using the supplementary information coming from the knowledge regarding the internal state along with observable outputs, there is almost infinite number of possible control schemes. Of course, the only limitation in this case is the controlability and maximum signal amplitudes which can be considered in the specified system. To fully benefit from the advances derived from the state-space framework, the state-based feedback in the following form

$$\mathbf{u}(k) = -\mathbf{K}\mathbf{x}(k) + \mathbf{v}(k), \tag{3.1}$$

where $\mathbf{v}(k)$ is the new reference input, can be introduced. In such a case, the system can be treated as a closed-loop one. The provided feedback can change system characteristics, therefore the closed-loop formula can be used in the following form

$$\mathbf{x}(k+1) = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}(k) + \mathbf{B}\mathbf{v}(k), \quad \mathbf{x}(0) = \mathbf{x}_0. \tag{3.2}$$

This expression shows the behavior of state equation. Naturally, the second state-space framework equation remains unchanged.

Of course two different scenarios should be considered here. In the unstable case the simulation horizon needs to be appointed arbitrarily as there is no reasonable restriction despite computational effort. However, in a stable case, the boundaries of meaningful calculations are clear, since after enough simulation steps the system reaches its steady-state. Of course, the eigenvalues connected with the stability of closed-loop plant can be calculated from

$$det(z\mathbf{I} - \mathbf{A}^*) = 0, \tag{3.3}$$

with the substitution $\mathbf{A}^* = \mathbf{A} - \mathbf{B}\mathbf{K}$.

It is clear, that the simulation horizon N is associated with the lowest power of matrix

$$(\mathbf{A}^*)^{N-1} = \mathbf{0}. \tag{3.4}$$

Of course, in the examined case the arbitrary sample of state vector can be calculated from the

$$\mathbf{x}(k) = (\mathbf{A}^*)^k \mathbf{x}(0), \tag{3.5}$$

for all $k < N$.

Naturally, the behavior of the closed-loop system can now be assigned almost arbitrarily. Poles, zeros, transient states and settling time of considered plant may be influenced with proper feedback. During past decades, a plethora of different control strategies have been developed. In the following chapter, some of them shall be shortly described in order to show characteristic peculiarities and differences between them.

3.1 State deadbeat control

The concept of proper linear feedback applied to the state-space systems is continuously subject to wide research scope. The natural succession of development in this area resulted in idea of placing all closed-loop poles at zero. With astonishing performance observed in terms of control speed, the deadbeat control became one of deeply developed frameworks. As result of different advanced studies, the traits of considered control strategy are obtained such as robustness, minimum settling time or even LQ optimality [48, 49].

With the opportunity to obtain different properties it seems even more interesting to take systems with more input than state variables into account. In such a case the matrix of static gain in the state-feedback framework can submit to multiple outer limitations, fulfilling the deadbeat requirements at the same time. For example, the deadbeat control can be enhanced in terms of robustness or control energy.

The deadbeat control can be achieved in several manners. The most common way is to solve a parameter matrix symbolic equation to obtain symbolic eigenvalues and then equaling them to zero.

The second possible technique devoted to placing all eigenvalues of closed-loop matrix at zero involves the already mentioned inverses. It has been shown that basing on the parameter matrices from the system description, there is a possibility to obtain the deadbeat properties [50]. In such a case the application of state feedback in form of

$$\mathbf{u}(k) = -\mathbf{B}^{-1}\mathbf{A}\mathbf{x}(k), \tag{3.6}$$

in the square matrix $\mathbf{B}$ scenario, or

$$\mathbf{u}(k) = -\mathbf{B}^{\mathrm{R}}\mathbf{A}\mathbf{x}(k), \tag{3.7}$$

when more inputs than state variables occurred is justified. It is worth mentioning that the requirement of $n \leq n_u$ needs to be fulfilled here. It is also worth mentioning that in the second nonsquare case there is a possibility to apply any nonsquare matrix right-inverse. However, the minimum-norm unique T-inverse is used most frequently, as it is correlated to the minimum-energy input property. Let us now switch to two corresponding simulation examples involving the new issues.

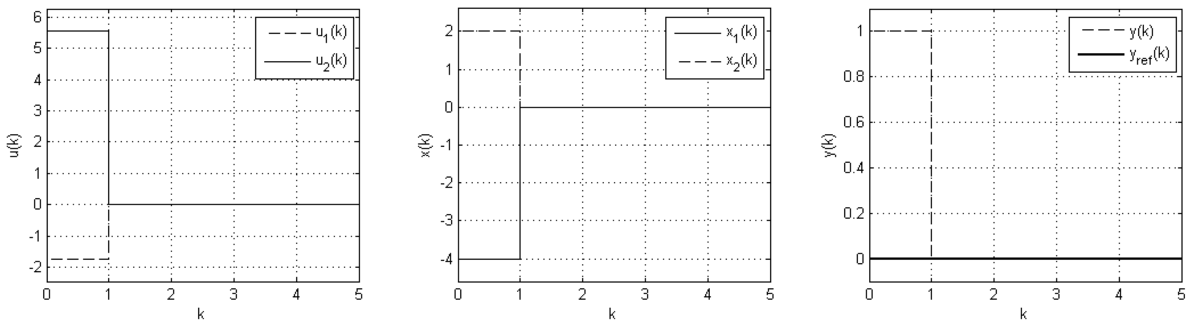


Figure 3.1: Control, state and output signals: Zero-state deadbeat control

The closed-loop plant with equivalent Jordan normal form $\mathbf{J} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ differs from the one obtained in the former deadbeat instance, thus the distinction can be performed in an analytical way.

3.2 Output deadbeat control

The output deadbeat control is yet another state-feedback-based control strategy [51–53]. Nevertheless, the objective is slightly different here. The minimization of control error and output settling time is rather prioritized over eigenvalue placement. In this context, the term 'deadbeat' refers to the steady-state of the output variables obtained in a less than arbitrary number of steps. Of course, the output settling time is expected be possibly low. In other words, the challenge present during the design of output deadbeat scheme is to find such a state feedback that will guarantee the following behavior

$$\mathbf{C}(\mathbf{A} + \mathbf{BK})^k \mathbf{x}(0) = \mathbf{0}, \forall \mathbf{x}(k), k \geq \gamma. \tag{3.8}$$

Of course, γ denotes the settling time, after which the output remains at the setpoint. However, such behavior is hard to obtain in an analytical way, thus another indicators shall be concerned. In the case, whenever it is possible, the mentioned behavior is related to the problem of locating all so-called output eigenvalues at zero. Let the output eigenvalues be defined as roots of the following expression

$$det(\mathbf{C}(\mathbf{A} + \mathbf{BK})) = \mathbf{0}. \tag{3.9}$$

The output deadbeat control is naturally obtained by finding proper state feedback, thus this control strategy is mentioned there. Of course this particular consideration is valid only if the matrix from Eq. (3.9) is a square one.

As it is stated, the main goal of output deadbeat control is to obtain the steady state of output variables in possibly no time. However, in opposite to the state deadbeat framework there is no such clear analytical indices that will facilitate the scientific effort.

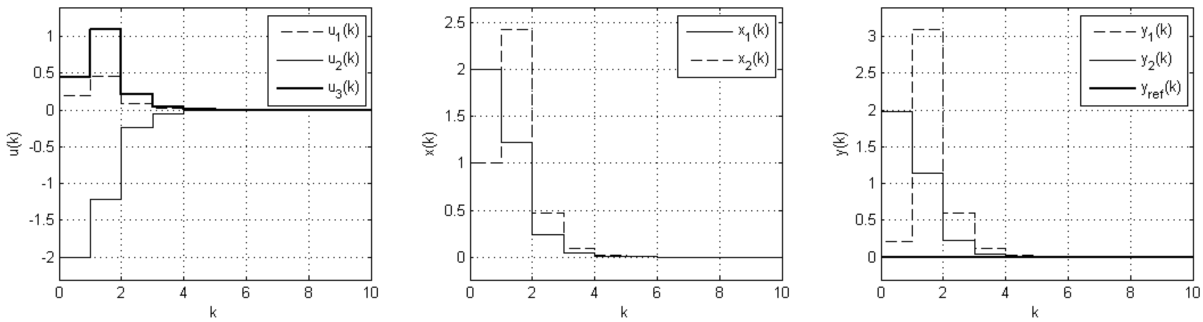


Figure 3.2: Control, state and output signals: Output deadbeat control

The output deadbeat properties can be also revealed with the output matrix

$$\mathbf{CA}^* = \begin{bmatrix} -0.7053 & 1.8962 \\ -0.2623 & 0.7053 \end{bmatrix},$$

with zero eigenvalues, so the output deadbeat control has just been established. However, this particular example does not reach the maximum possible speed in terms of obtained output steady state. Therefore, it is clear, that with feedback better than in above scenario there is a possibility to improve this control strategy.

Moreover, as in the classical state deadbeat there is not only possibility to obtain zero eigenvalues, but also entire considered matrix equal to a zero matrix. Nevertheless, this case is a different self-sufficient research area, called the perfect control, described in the next section.

3.3 Unstable/stable-pole perfect control

Perfect control is one of the most demanding control strategies. The main trait of this control strategy is the lowest obtainable control error, as it disappears right after delay derived from system description. This behavior can be achieved by minimization of proper performance index as follows

$$e = \sum_{k=0}^{N-1} ((\mathbf{y}_{\mathrm{ref}}(k) - \mathbf{y}(k))^{\mathrm{T}} (\mathbf{y}_{\mathrm{ref}}(k) - \mathbf{y}(k)). \tag{3.10}$$

In the case of systems with delay $d = 1$ this property can be obtained after equating the one-step deterministic predictor $\mathbf{y}_{\mathrm{ref}}(k+1)$ to the setpoint as follows

$$\mathbf{CAx}(k) + \mathbf{CBu}(k) = \mathbf{y}_{\mathrm{ref}}(k+1). \tag{3.11}$$

From the above-presented predictor the perfect control formula for LTI MIMO discrete-time state-space systems can be calculated. After some manipulation, the control law sounds in the following form

$$\mathbf{u}(k) = (\mathbf{CB})^{\mathrm{R}}[\mathbf{y_{ref}}(k+1) - \mathbf{CAx}(k)], \tag{3.12}$$

where $(\mathbf{CB})^{\mathrm{R}}$ stands, e.g., for unique minimum-norm right T-inverse of considered matrix. It is clear that the perfect control fits into state-feedback expression with

$$\mathbf{K} = (\mathbf{CB})_0^{\mathrm{R}}\mathbf{CA}. \tag{3.13}$$

In this scenario, the poles rely on the numerical solution of nonunique inverse $(\mathbf{CB})^{\mathrm{R}}$. After application of the minimum norm right T-inverse we arrive at the control zeros of first type connected with the stability of IMC-based plant, such as perfect control [43]. Of course, the unstable instances are not considered here, as the usage of different nonunique inverses is a remedy to this event. The proper mathematical effort can guarantee that the stability can be established, unless there are unstable transmission zeros, if any in the nonsquare case [43,45,54]. The characteristic behavior of perfect control systems can be observed in the exemplary instance shown below.

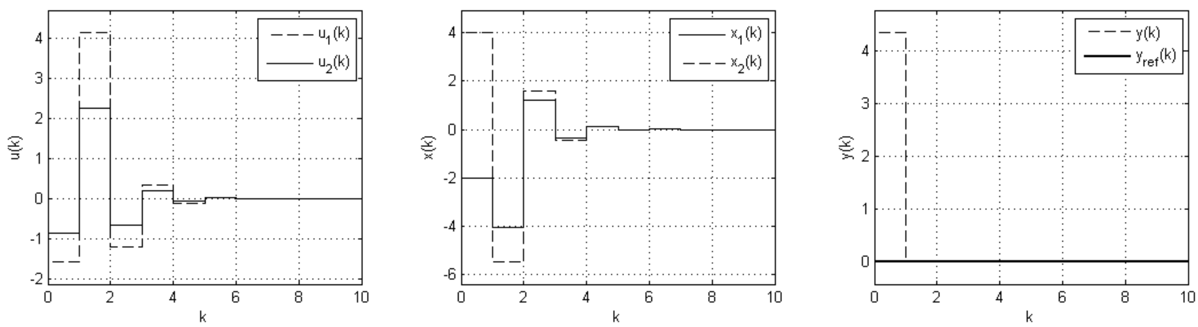


Figure 3.3: Control, state and output signals: Stable-pole perfect control

Interestingly, in perfect control instances the obtained output eigenvalues are equal to zero, same as the whole output matrix $\mathbf{CA}^* = \begin{bmatrix} 0 & 0 \end{bmatrix}$. It is clear now, that perfect control is indeed a special case of output deadbeat control with possibly fastest output stabilization. In contrast to the deadbeat control from Eq. (3.7), the perfect control simultaneously preserves the nonzero internal dynamics of perfect control plant.

Summing up, the perfect control algorithm focuses on the setpoint of output variables without any compromises dealing with any other performance indices.

Chapter 4

Pole-free perfect control

Among different state-feedback-based control systems discussed in the previous section, there is yet at least one strategy worth of scientific consideration. As it has been shown, perfect control systems can have different properties. The settling time, sometimes referred to as control speed, is usually connected to system poles. Thus during the pursuit of the maximum-speed perfect control the idea is to place all closed-loop control system poles at zero.

The properties connected with zero eigenvalues have already been established for MV control [43]. Nevertheless, in this stochastic MV case it has been done in a heuristic simple way by minimization of sum of absolute poles value. The recent studies and developed mathematical background allow to finally present the whole subject in a comprehensive study.

For now, the arbitrary closed-loop properties were obtained with application of proper state feedback. In the perfect control scenario the linear gain matrix at first needs to submit to the following framework

$$\mathbf{K} = (\mathbf{CB})^{\mathrm{R}}\mathbf{CA}, \tag{4.1}$$

where in this case $(\mathbf{CB})^{\mathrm{R}}$ denotes any right inverse of $\mathbf{CB}$. Now, in the pole-free consideration the main target is to find such form of any unique/nonunique matrix inverse that will guarantee the pole-free behavior. It is clear, that the designation 'pole-free' should be understood in terms of all closed-loop control system poles placed exactly at zero.

Since the nonunique matrix inverses are involved, the second type control zeros should be considered here (see Ref. [43]). In this scenario, the closed-loop poles of inverse system (equivalent with control zeros) can be placed almost arbitrarily. The graphical representation of proper pole-freeing procedure is shown in the figure (4.1).

With the desired zero-location of perfect control closed-loop poles, the system is expected to obtain possibly lowest settling time and maximum accuracy. Being one of a kind in connection of deadbeat and perfect control strategies, a plethora of intriguing properties are expected and welcome. The number of (non)zero eigenvalues obtained under the minimum-norm inverse or the lowest obtainable steady-state might be a subject of comparison study. Naturally, with different limitations imposed by considered system, the obtained results can vary within simulation examples.

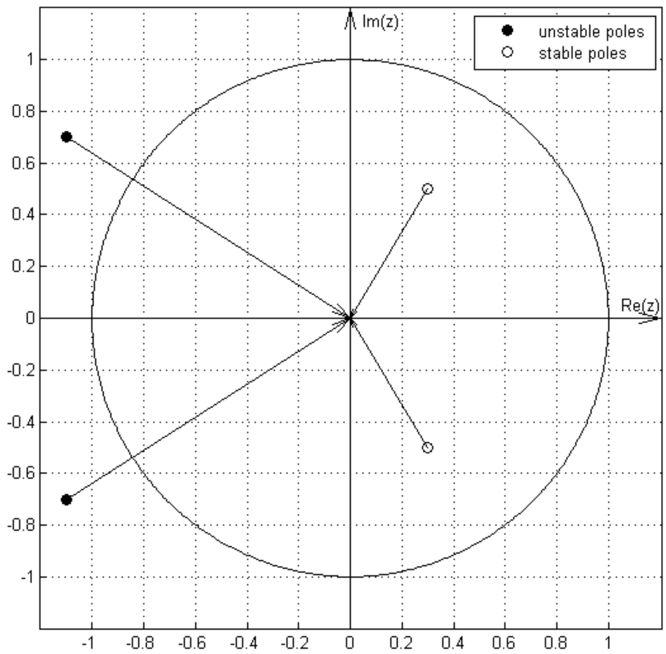


Figure 4.1: Idea of pole-free placement

Of course, the pole-free manner, same as a stable-pole one, implies a number of requirements that need to be fulfilled both before and during control design as well as a plethora of numerical and structural manners. These criteria are presented in the upcoming section.

4.1 Pole-free perfect control requirements

As in almost every control instance there are some requirements that need to be fulfilled to ensure that the pole-free perfect control is achievable. At the beginning, the assumptions of controlability and observability need to be made. In such an extreme control strategy both of these properties need to be ensured, so the state and output variables can be arbitrarily moved from chosen set of values to another one. The controlability is especially crucial, as all of eigenvalues need to be managed by pole-placement method. Moreover, matrices consisting the system description need also to be of proper dimensions in order to enable the perfect control development. It is well known, that the perfect control is valid only for right-invertible systems, so plants with input number equal or larger than number output variables with additional requirement that matrix **CB** if of full rank should be examined.

Initially, the pole-free perfect control was connected to the deadbeat-like solution

$$(\mathbf{CB})^{\mathrm{R}} = \mathbf{B}^{\mathrm{R}}\mathbf{C}^{\mathrm{L}}, \tag{4.2}$$

resulting in zero-closed-loop system matrix end up as follows

$$\mathbf{A} - \mathbf{B}(\mathbf{CB})^{\mathrm{R}}\mathbf{CA} = \mathbf{A} - \mathbf{BB}^{\mathrm{R}}\mathbf{C}^{\mathrm{L}}\mathbf{CA}. \tag{4.3}$$

However, the Eq. (4.2) is only valid in the case of plants with square matrix $\mathbf{C}$, all supported by zero control system eigenvalues.

Remark 4.1 *It is worth mentioning, that in the case with square matrix* $\mathbf{C}$ *the perfect control formula reduces to the deadbeat law as in Eq. (3.7). Thus, not only the same eigenvalues are obtained, but also control and state signals are identical.*

Naturally, the requirements needed to obtain the pole-free perfect control are covering those from stable-pole case, so far. However, some additional limitations are needed here. During the design of pole-free perfect control it is necessary to influence eigenvalues of the closed-loop matrix. Thus, the number of independent degrees of freedom needs to be at least equal to number of linearly independent eigenvalues that are equated to zero. This property cannot be obtained during the proper process of control design, as the size of matrix consisting degrees of freedom is strictly connected to the sizes of matrices of state-space description.

It is worth mentioning that the system matrix $\mathbf{A}$ is not necessarily needed to be of full rank. This is supported by fact, that the target is to reduce the number of non-zero eigenvalues, so any rank-reducing factors are not strictly forbidden. However, the mentioned system still needs to be controllable. In particular this requirement applies to non-zero mods that will disable the pole-free perfect control design. As the rank of closed-loop system matrix is related to the number of its unique eigenvalues, the open-loop system matrix shall adhere to the following relation

$$rank(\mathbf{A}) \geq n_y. \tag{4.4}$$

In the minimum-norm perfect control design the closed-loop system rank is usually connected with number of system outputs, so this also shows, that the closed-loop system matrix will be of maximum n_y-rank.

Additional requirement can be included after considering the already mentioned Jordan normal form derived from Eq. (2.27) as follows

$$tr(J(\mathbf{A}^*)) = tr(\mathbf{P}^{-1}\mathbf{A}^*\mathbf{P}). \tag{4.5}$$

Thus, having all eigenvalues on the main diagonal, the pole-free requirement suddenly clarifies. As the target is to obtain zero eigenvalues it is clear that the trace of closed-loop matrix submits to the following equation

$$tr(\mathbf{A}^*) = 0. \tag{4.6}$$

The last pole-free perfect control requirement is based on the Cayley-Hamilton theorem. It was shown, that every square matrix fulfills its characteristic polynomial being monic polynomial. In order to place all poles at zero it is necessary to shape the characteristic polynomial to have only zero parameters beside its highest power-based factor. This will ensure, that the only eigenvalue is multiple zero root of characteristic equation. To obtain this pole-free property, the subsequent expression should hold

$$(\mathbf{A}^*)^n = \mathbf{0}. \tag{4.7}$$

Of course, the number of linear independent parameters consisting the degrees of freedom needs to be at least equal to the number of poles that need to be placed.

In order to greater diversification of pole-free and stable-pole perfect control scenarios, the crucial differences are given in the next section.

4.2 Pole-free vs. stable-pole perfect control

In this section, a study showing the distinction between different perfect control manners is shown. The different properties enabled by proper pole-placement method applied to the perfect control manner can result in advancement in terms of speed, control energy, robustness or stability. To properly compare two different perfect instances some performance indices, other than in Eq. (3.10), need to be defined. For this reason the two versions, unique and nonunique ones, will be compared in terms of settling time and control energy. The mentioned control energy will be considered as

$$E = \sum_{k=0}^{N-1} \mathbf{u}^2(k), \tag{4.8}$$

where $\mathbf{u}^2(k)$ denotes the vector of squared control values obtained during simulation run. With such a performance index, it is possible to compare pole-free and stable-pole instances in the context of control energies.

During the study, the control energy will be denoted twofold. In the case of minimum-norm right T-inverse applications, the obtained energy will be called E_{SP} as related to stable nonzero poles. On the other hand, the perfect control energy of instances covering the pole-free-based behavior will be named E_{PF}.

It is also worth mentioning that the perfect control scenarios can also be differentiated in terms of other than control energy derived from the potential of input variables.

Example:

Consider again a second-order 3-input system described by the following matrices $\mathbf{A} = \begin{bmatrix} -0.3700 & 1.4500 \\ -1.9200 & 0.8000 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} -0.4500 & 0.9000 & -1.5000 \\ 1.4000 & -0.8200 & 0.1000 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} -0.2000 & 0.4000 \end{bmatrix}$, with initial condition $\mathbf{x}_0^{\mathrm{T}} = \begin{bmatrix} -3 & 6 \end{bmatrix}$. With the minimum-norm right T-inverse applied into perfect control design the following closed-loop system matrix is obtained as $\mathbf{A}^* = \begin{bmatrix} -1.4681 & 1.4975 \\ -0.7340 & 0.7487 \end{bmatrix}$. The signals with control energy $E_{SP} = 221.5384$ are presented below.

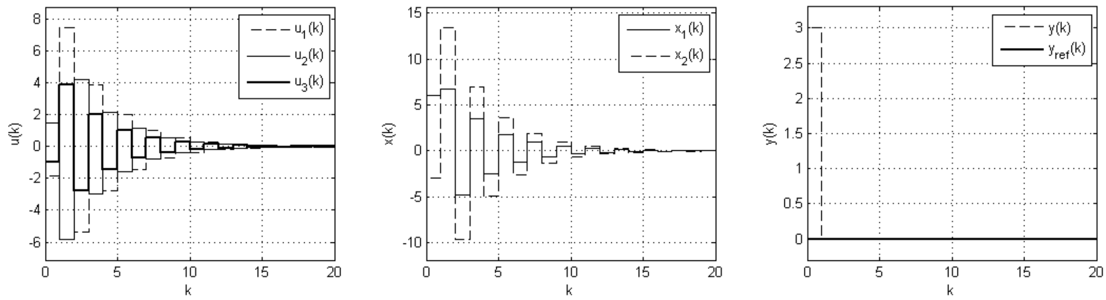


Figure 4.2: Control, state and output signals: Stable-pole T-inverse-based instance

On the other hand, with the application of right σ-inverse with attached degrees of freedom $\beta = \begin{bmatrix} 13.3690 \\ 1.1607 \\ -0.5569 \end{bmatrix}$, the following state feedback is obtained $\mathbf{K} = \begin{bmatrix} -1.1728 & 0.0507 \\ -0.1018 & 0.0044 \\ 0.0489 & -0.0021 \end{bmatrix}$. The pole-free perfect control system is now described by the nilpotent closed-loop matrix $\mathbf{A}^* = \begin{bmatrix} -0.7328 & 1.4657 \\ -0.3664 & 0.7328 \end{bmatrix}$ with its Jordan normal form $J(\mathbf{A}^*) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Signals obtained during the simulation run are presented in the chart below.

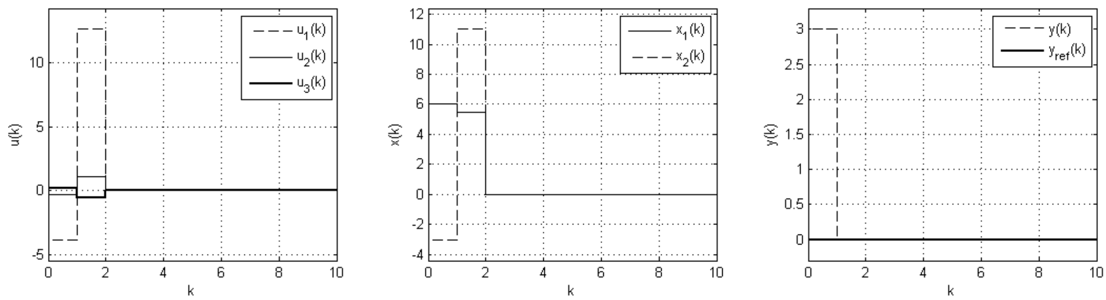


Figure 4.3: Control, state and output signals: Pole-free σ-inverse-based instance

In this pole-free instance the control energy is equal to $E_{PF} = 175.3333$. With $E_{\mathrm{PF}} < E_{\mathrm{SP}}$ this example can be a basis for more advanced energy-based research, done later in this thesis.

4.3 Special MISO second-order case using σ-inverse

The general case of pole-free perfect design requires some sophisticated calculation considering symbolic variables representing the feedback matrix. Such approach is usually used throughout this thesis to obtain proper degrees of freedom.

Nevertheless, it has been observed that there are some special cases in which this calculation effort can be noticeably limited. Using the characteristic equation as follows

$$det(\mathbf{A} - \mathbf{B}(\beta)^{\mathrm{T}}((\mathbf{C}\mathbf{B}\beta^{\mathrm{T}})^{-1}\mathbf{C}\mathbf{A}) = 0. \tag{4.9}$$

with symbolic degrees of freedom β, further calculations can be done. Using the reverse symbolic calculation, the noticeable development was performed recently. As a result of this study the following formula

$$\boldsymbol{\Xi}\boldsymbol{\Theta}\mathbf{B}\beta^{\mathrm{T}} = 0, \tag{4.10}$$

where $\boldsymbol{\Theta} = \begin{bmatrix} -\mathbf{A}(2,1) & \mathbf{A}(1,1) \\ -\mathbf{A}(2,2) & \mathbf{A}(1,2) \end{bmatrix}$ and $\boldsymbol{\Xi} = \begin{bmatrix} -\mathbf{C}(1,2) & \mathbf{C}(1,1) \end{bmatrix}$, was originally postulated (see Ref. [25]). Preserved β, used in the perfect control design, locating all of the closed-loop perfect control poles at zero.

Example:

Consider a nonsquare right-invertible second-order system with the open-loop system matrix $\mathbf{A} = \begin{bmatrix} 0.2000 & 0.4000 \\ -0.1000 & 0.3000 \end{bmatrix}$, input matrix $\mathbf{B} = \begin{bmatrix} 0.7000 & 0.1000 \\ -0.2000 & -0.8000 \end{bmatrix}$ and $\mathbf{C}^{\mathrm{T}} = \begin{bmatrix} 1 & 2 \end{bmatrix}$. In this scenario the initial conditions can be any. Thus we have

$$\begin{bmatrix} -2 & 1 \end{bmatrix} \begin{bmatrix} 0.1000 & 0.2000 \\ -0.3000 & 0.4000 \end{bmatrix} \begin{bmatrix} 0.7000 & 0.1000 \\ -0.2000 & -0.8000 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = 0.$$

For $\begin{bmatrix} -0.3500 & -0.0500 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = 0$ we arrive at $\beta = \begin{bmatrix} 2.8571 & -20.0000 \end{bmatrix}$.

Of course, the σ-inverse has no exclusiveness in terms of obtainable pole-free perfect control. Thus the pole-free instance with use of nonunique right H-inverse is presented in the next section.

4.4 Application of nonunique H-inverse in the pole-free perfect control design

In the previous section it was shown that the pole-free perfect control can be obtained using the right σ-inverse. The proper degree(s) of freedom calculated in dedication for given system result in pole-freeing perfect feedback. Naturally, the σ-inverse in not the only available tool for pursuing pole-free perfect control. Identical results can be obtained with the application of the right H-inverse of Eq. (2.19). The subject of this section has already been applied to the perfect control design, however the pole-freeing action was rather overlooked. Thus, the H-inverse is used here to obtain the proper closed-loop plant.

Example:

Consider a second-order LTI state-space system described with the following matrices $\mathbf{A} = \begin{bmatrix} -0.6300 & 1.0300 \\ 0.5900 & 1.3500 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0.2000 & 0.6800 \\ 1.6700 & 1.3400 \end{bmatrix}$, $\mathbf{C}^{\mathrm{T}} = \begin{bmatrix} 0.6500 \\ -0.8700 \end{bmatrix}$ and $\mathbf{x}_0^{\mathrm{T}} = \begin{bmatrix} 7 - 2 \end{bmatrix}$. In this case the minimum-norm H-inverse results in a single nonzero eigenvalue. It is worth mentioning, that in this instance the number of degrees of freedom is lower than in the case of σ-inverse application. By usage of the H-inverse with $L = 1.476$, we obtain the following right inverse $(\mathbf{CB})^{\mathrm{R}} = \begin{bmatrix} 1.8654 \\ 0.2187 \end{bmatrix}$, which results in linear gain $\mathbf{K} = \begin{bmatrix} -0.0050 & -0.0337 \\ 0.0743 & 0.5050 \end{bmatrix}$. Such state-feedback perfect control plant behavior provides the signals depicted in figure below.

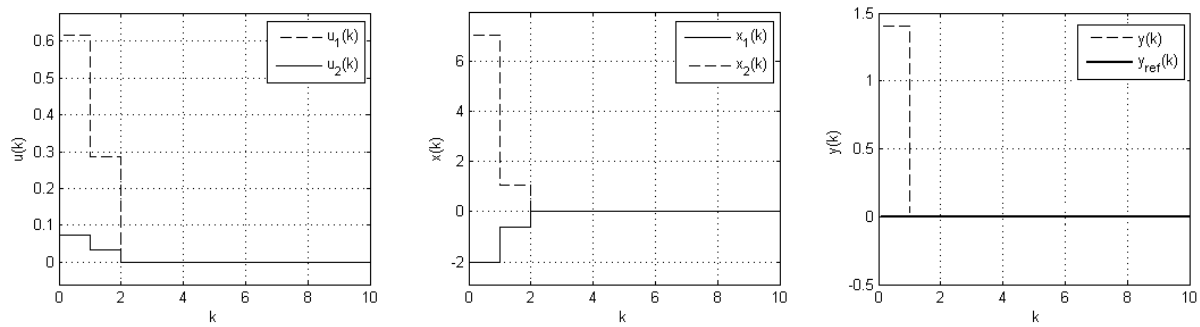


Figure 4.4: Control, state and output signals: Pole-free H-inverse application

The anticipated system response is obtained here, thus the possibility to achieve pole-free perfect control with H-inverse employment is now not as issue.

However, all of above-presented cases have one trait in common. The zero reference value imposed on the system entails the zero-steady states of both control and state variables. A study covering an extension of pole-free formulas to the nonzero setpoint plants is briefed in the next section.

4.5 Zero vs. nonzero reference value

The discussed pole-free perfect control instances shown so far, that the zero-steady state can be achieved in almost no time. However, in plethora of applications the references/setpoints of output variables constitute set of values other than zero, in general. Of course, the arbitrary output values can be reached since the system is said to be controllable. Thus, a brief study over perfect control plants with nonzero reference is conducted here. Naturally, the pole-free perfect control obtained for a given system can be enriched with the consideration of reference signal. In such case, the perfect control formula is related to the following form

$$\mathbf{u}(k) = (\mathbf{CB})^{\mathrm{R}}_{\mathrm{PF}}(\mathbf{y_{ref}}(k+1) - \mathbf{CAx}(k)), \tag{4.11}$$

where $(\mathbf{CB})^{\mathrm{R}}_{\mathrm{PF}}$ stands for such inverse that places all closed-loop poles at zero. The main difference between zero and non-zero reference system is a preamplifier applied to the setpoint values. A static inverse-based gain ensures proper system output behavior. Of course, the dynamics of control system is not influenced by the arbitrary reference value.

The main issue, raised during this consideration, is the simulation horizon. For zero-reference systems it was clear that the simulation can be terminated after reaching zero steady-state values. However in the nonzero case, there are no such clear boundaries since some performance indices, e.g. control energy, are dependent on simulation time. Therefore, an arbitrary time limitation can be imposed to the simulation runs. Crucially, same limits can be applied to both of compared instances. Since the energy-based performances together with the influence of the mentioned limitations are presented later in the thesis, let us now continue with a simple simulation example. A comparison between zero and nonzero pole-free perfect control instances is presented below.

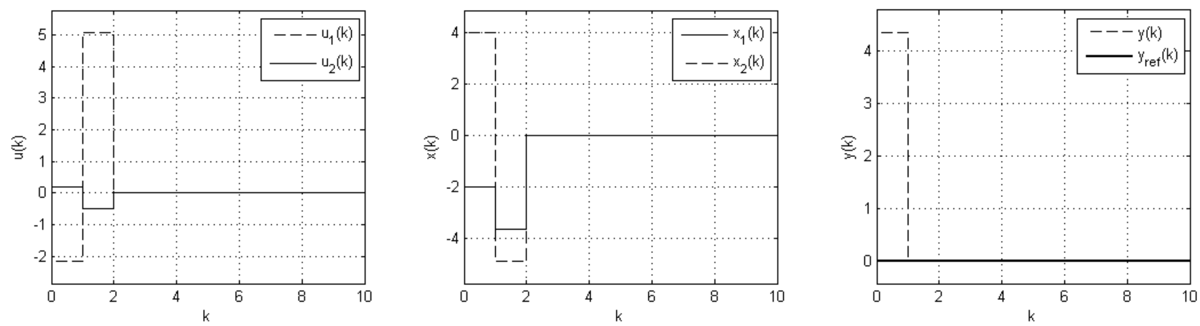


Figure 4.5: Control, state and output signals: Pole-free instance, $y_{\mathrm{ref}} = 0$

On the other hand, for the same system but with non-zero setpoint slightly different signals were obtained. The main achievement here is a fact, that any arbitrary output value can be obtained in the same number of samples as in the zero-reference scenario. The corresponding signals are presented below.

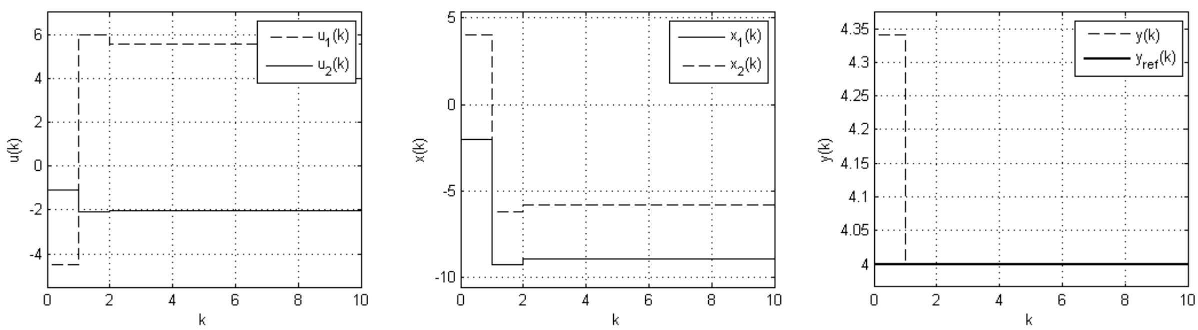


Figure 4.6: Control, state and output signals: Pole-free instance, $y_{\mathrm{ref}} \neq 0$

As it is shown, both during zero and nonzero setpoint studies, some interesting energy properties are revealed. Despite obvious advantages in the speed and settling time, the complex study over the widely-understood energy performance is presented in the next chapter.

4.6 Summary

The pole-free perfect control phenomena were presented in this section. First, the requirements that are needed to be fulfilled. These requirements are considering the properties of controlled state-space system. In general, pole-free perfect control is not obtainable without meeting those pre-requirements. However, with the assumption of attainable pole-free schemes, the additional formulas were presented, allowing to place all closed-loop perfect control poles exactly at zero. The well known rules were applied in order to obtain such linear state-based feedback that results in a proper closed-loop system matrix.

It is also shown, that all mentioned machinery can be supplied with the use of nonunique inverses of nonsquare matrices. In particular, the application of right H- and σ-inverse are subject of this study. Moreover, the case covering the right σ-inverse is supported by interesting numerical procedure. A symbolic calculation together with matrix-based study led to synthesis of formula, that allowed to obtain proper (in terms of pole-freeing action) degree of freedom β. Such procedure enables, without laborious calculations, to perform a number of different simulation tests providing pole-free perfect control properties.

The simulations of pole-free perfect control instances revealed the deadbeat-like characteristics. With steady-state obtained in almost no time and high signal amplitudes these two control strategies can be mistaken. Moreover, both of control frameworks have all its eigenvalues equal to zero, thus additional differentiation was needed to be made. Fortunately, the Jordan normal form of closed-loop system matrices constitutes clear determinant and distinction between deadbeat and pole-free perfect control systems.

Additionally, a study on other then zero reference plant is conducted. With the adjective static matrix gain applied to the output reference signal, the pole-free perfect control for arbitrary reference values is obtained. Anyway, it is clear, that in cases covering nonzero reference, the control signals can be divided into two parts, dealing with system dynamics and reference value, respectively. Of course, the pole-free perfect control sustains its properties for the arbitrary setpoint of control system output vector.

Moreover, some interesting energy-based features occurred. The disclosure of instances, where the pole-free approach results in lower control energy than in the case of minimum-norm inverse applied to the perfect control system design, was a premise to a more complex energy-based study. The research studies covering an attempt to prejudge the energy outcome of perfect control scenario are presented in the next section.

Chapter 5

Energy performance

5.1 Energy of state variables

The state signals together with their energy were subject of different studies over past years. Numerous papers have considered the state variables in the terms of stability, robustness, control speed and energy efficiency. Throughout this section the main interest is focused on the state variability and related state energy. Therefore, the energy of state variables can be calculated with the following formula

$$E_x = \sum_{k=0}^{N-1} \mathbf{x}^2(k). \tag{5.1}$$

Of course, in the case of considered vectors this formula can be presented in an equivalent form as follows

$$E_x = tr(\mathbf{x}^{\mathrm{T}}(k)\mathbf{x}(k)). \tag{5.2}$$

There are also some interesting properties that can be obtained by application of well known relations into this consideration. Moreover, in case of positive systems, i.e. systems with signals having only non-negative values, the energy of state variables is in obvious relation with the sum of state signals. The mentioned sum can be obtained with the following formula

$$X = \sum_{k=0}^{N-1} \mathbf{x}(k). \tag{5.3}$$

Since the subsequent samples of state variable are successive in terms of geometric series, the following equation can also be obtained with [55]

$$X = X_0 + \frac{X_1}{1-\lambda}, \tag{5.4}$$

where λ is the only non-zero closed-loop eigenvalue. This eigenvalue, here $\lambda < 1$ can be treated as the decrement of suppression. Moreover X_0 and X_1 are energies of respective state samples.

Thus, as originally presented in Ref. [29], the norm-based index as follows should be recalled

$$N_{\mathrm{x}} = ||\mathbf{A}^{*}\mathbf{x}_0||. \tag{5.5}$$

This index can be used in order to anticipate the energy of state vector. Of course, this index does not fully cover all state-feedback systems, but the useful behavior of presented equation is depicted in the simulation example made below.

Example:

Consider a second-order LTI state-space system described with the following matrices $\mathbf{A} = \begin{bmatrix} -0.7800 & -0.3400 \\ 1.0300 & -0.2700 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1.0400 & 0.2800 \\ -0.3000 & 0.2000 \end{bmatrix}$, $\mathbf{C}^{\mathrm{T}} = \begin{bmatrix} 0.5500 - 0.7100 \end{bmatrix}$. In this experiment three different scenarios were considered. Two of them being the perfect control instances with application of right T-inverse and pole-free σ-inverse, respectively. However, the greatest part of numerous simulation runs is constituted by perfect control simulations with the use of randomly generated degrees of freedom β of right σ-inverse. The obtained results are presented below.

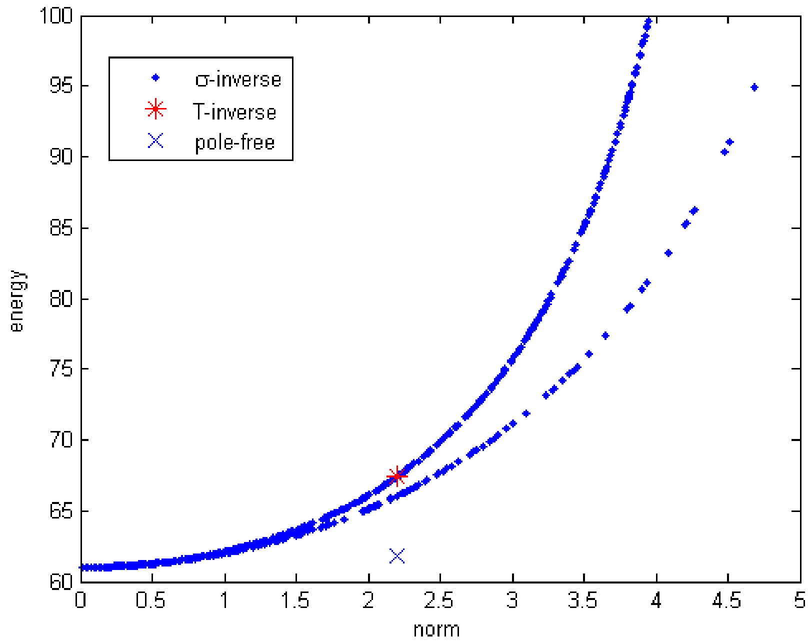


Figure 5.1: Energy-based norm vs. energy of state variables

It is clear now, that the minimization of presented index seems to be a reasonable action. However, the energy within state variables is not strictly connected to the control energy. Thus, the issue concerning the energy of control signals is shown in the next section.

5.2 Control energy of zero-reference systems

Optimization of systems using performance indices based on the amount of energy fed into the control system is an actual scientific topic. Widespread application of energy efficient solutions affects almost every aspect of modern daily life, including transportation, construction technology, household goods and environmental protection. In the industrial application of energy saving methodology the substantial improvement can be achieved both in software and hardware layers. The development in lowering of consumed energy is often connected with material engineering which allows to go beyond previously established boundaries. Same in perfect control design, the control signals can be optimized in the context of consumed energy [56]. As the simple examples from Section 4.2 have shown, the energy-related performance can be unintuitive since in some cases the pole-free instances result in both higher speed and lower control cost. Thus an attempt to prejudge the control energy is presented here.

The summarized energy of control signals has already been mentioned. Nevertheless, the definition from Eq. (4.8) is equivalent to the following form

$$E = \sum_{k=0}^{N-1} ||\mathbf{u}(k)||^2. \tag{5.6}$$

From the definition of matrix norm the energy can be also calculated according to the formula as follows

$$E = \sum_{k=0}^{N-1} \langle \mathbf{u}^{\mathrm{T}}(k), \mathbf{u}(k) \rangle. \tag{5.7}$$

Now, in the considered case of zero-reference state-feedback control it is clear, that we can rewrite this relation to the following form [29]

$$E = \sum_{k=0}^{N-1} tr(\mathbf{K}^{\mathrm{T}}\mathbf{K}\mathbf{x}(k)\mathbf{x}^{\mathrm{T}}(k)). \tag{5.8}$$

It is worth mentioning, that the trace is invariant under cyclic permutation, which allows to group similar terms together. As it was already shown, the arbitrary sample of state vector can be calculated from initial conditions and proper power of closed-loop system matrix $\mathbf{A}^*$. Thus the following equation holds

$$E = \sum_{k=0}^{N-1} tr(\mathbf{K}^{\mathrm{T}}\mathbf{K}(\mathbf{A}^*)^k\mathbf{x}(0)((\mathbf{A}^*)^k\mathbf{x}(0))^{\mathrm{T}}), \tag{5.9}$$

constituting a basis for further energy-based considerations.

Some conclusions were drawn from the study originally presented in Ref. [28]. The final form of performance index from the mentioned paper is as follows

$$N_1 = ||\mathbf{K}^{\mathrm{T}}\mathbf{K}||\,||(\mathbf{A}^*)\mathbf{x}_0\mathbf{x}_0^{\mathrm{T}}(\mathbf{A}^*)^{\mathrm{T}}||. \tag{5.10}$$

Nevertheless, the unpublished yet recent study shows that the weighted sum, instead of classic one, can also be useful in the pursuit of possibly low control energy. In this way we obtain

$$N_2 = ||\mathbf{K}^{\mathrm{T}}\mathbf{K}||\,||e^{\mathbf{A}^*}\mathbf{x}_0\mathbf{x}_0^{\mathrm{T}}e^{\mathbf{A}^{*\mathrm{T}}}||. \tag{5.11}$$

This performance index will be used in order to determine if the energy of pole-free or stable-pole instances is expected to result in lower perfect control energy.

Some simulation experiments were performed. and similarly to the previous, state-based consideration, three different instances were considered to the perfect control design, i.e. pole-free, T- and random σ-inverse derived approaches.

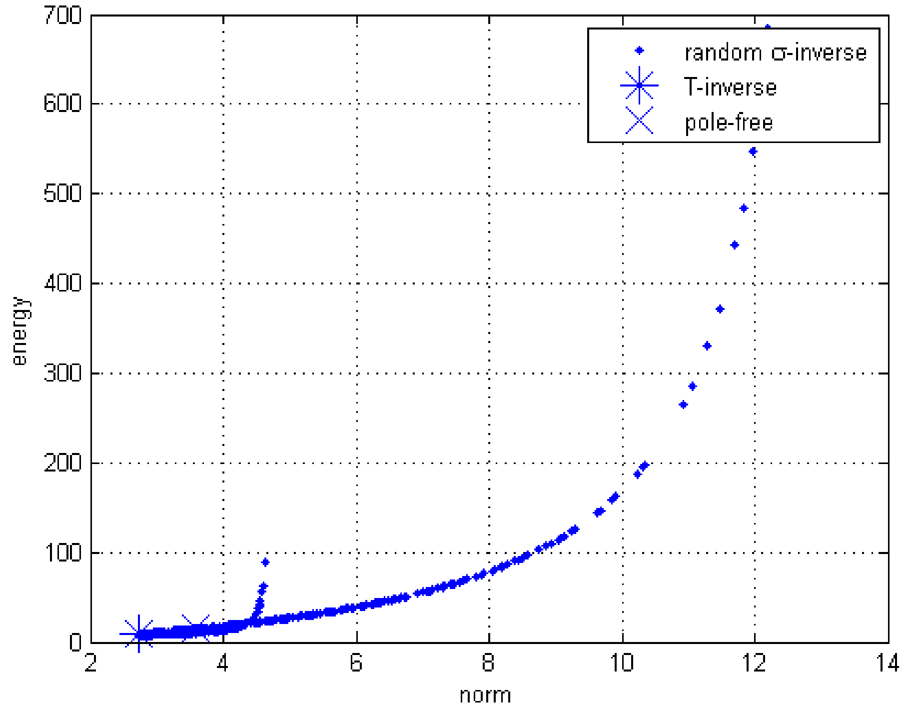
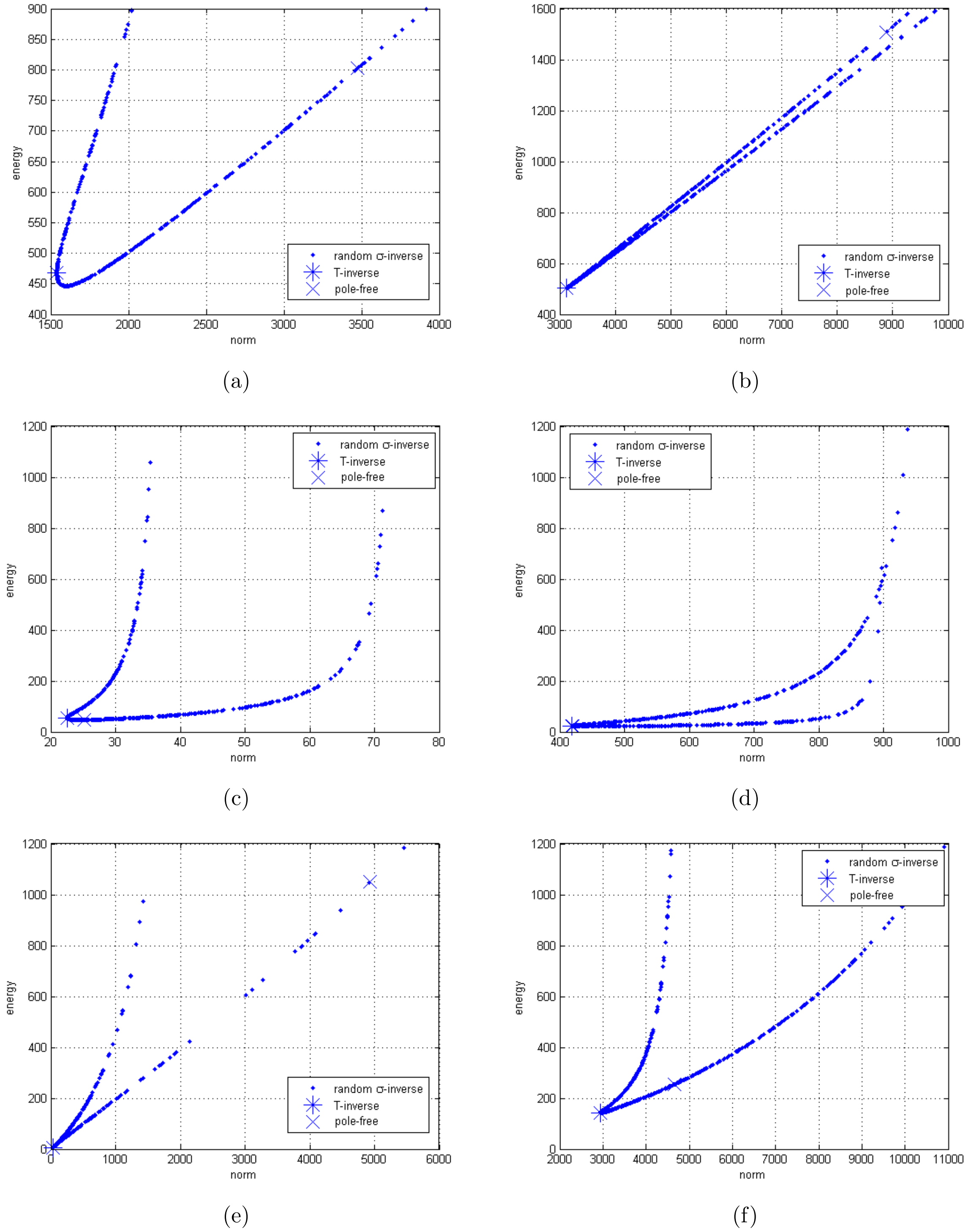


Figure 5.2: Control energy-based norm vs. control energy

Of course, the presented framework does not cover all possible cases. However, the shape of this characteristic is not the same for all systems, but justifies the minimization of the considered performance index in context of control energy optimization. A number of examples with different characteristics are presented below.



(a) (b) (c) (d) (e) (f)

Figure 5.3: Overview of different energy-based characteristics

5.3 Summary

The energy-based consideration is presented within this chapter. At the beginning a basic information about energy-based optimization were presented. Then a short review of energy of state variables was performed. With an application of the analytic performance index it was shown, that the energy stored within the state variables can be effectively minimized. A single simulation instance confirmed, that the presented approach leads to possibly low state energy.

However, more interesting seems to be a study covering energy of control signals, also presented in this chapter. A study with deep analytical foundations led to formulation of two performance indices that were successfully applied to the perfect control design. It is shown, that minimization of presented final index N_2 can result in lower control strategy. The use of complex matrix calculation allowed to shape the energy characteristic to have single region containing the minimum-energy point.

Therefore, the mentioned analytical indices can be used during comparison of pole-free and stable-pole perfect control instances. With simple, obtained with almost no computational effort, index the process of proper perfect control design can be significantly shortened. Without burdensome simulation effort the control energy of pole-free perfect control can automatically be predicted. With comparison of values of proposed indices there is a wide range of inverses for which the indices result in lower control energy. Therefore, the usage of formulas presented in this chapter seems to have wide area of application. Nevertheless, the proposed equation (5.11) does not cover all of possible cases, thus this topic is yet not closed.

Additionally, a study considering perfect control systems with non-zero reference was performed here. In contrast to previous study the steady state of control signals are the main focus here. In some manner the admission of non-zero reference system simplifies the whole issue. As the energy of steady control signals depends only on the inverse applied to the reference input values, the main determinant will be the norm of mentioned inverse. Of course, the whole consideration of zero-referenced systems can also be used here to minimize the control energy. The sufficient requirement is that the steady-states of compared control signals have the same amplitude, respectively, thus the energy of transient states are crucial again.

Summing up, a wide study considering control and state energy of perfect control systems was performed here. A useful tool was given with possible application in determining which approach, pole-free or stable-pole, will result in lower control energy. Nevertheless, there are still many yet unsolved issues, such as finding minimum-energy perfect control.

Chapter 6

Conclusions

A solution of pole-free perfect control problem is given in this thesis. Moreover, the inverses of nonsquare matrices with infinite number of possible degrees of freedom guarantee that the pole-free property can be obtained for considered right-invertible LTI MIMO state-space systems. The undertaken study of state-feedback-based perfect control strategy shows that the mentioned law can be enhanced with the application of proper inverse into the control design process. It is shown in the Ph.D. dissertation that the zero closed-loop pole location enabled by mentioned nonunique inverses results in measurable merits. After fulfilling the requirements presented throughout the work, the properties connected to higher control speed and stability are available.

The pole-free perfect control strategy obtained for a class of nonsquare systems also yields an improvement in the context of the summarized energy fed into the plant during the control action. The energy-related performance indices are presented in order to facilitate the minimum-energy perfect control design. With the analytical formula, the possibility to anticipate which degrees of freedom, being a part of closed-loop system matrix, results in lower control cost. It is worth mentioning that there are a plethora of pole-free cases with better energy performance than the stable-pole instances. This behavior is rather unexpected as it overthrows the usual compromise between control speed/accuracy and energy. Advanced work considering these peculiarities is expected in the near future.

The study presented in this thesis naturally can be a foundation for further considerations. For example, the extension of pole-free approach to time-varying or nonlinear plants still poses significant scientific challenge. Moreover, a universal formula allowing to numerically obtain the pole-free perfect control for specified inverses seems to be worthy of effort.

Summing up, the pole-free perfect control condensed in this thesis combines multiple complex branches of math and control engineering to obtain a possibly fast, cheap and accurate control strategy.

Bibliography

[1] S. Bańka and P. Dworak, "Efficient algorithm for designing multipurpose control systems for invertible and right-invertible MIMO LTI plants," *Bulletin of the Polish Academy of Sciences - Technical Sciences*, vol. 54, no. 4, pp. 429–436, 2006.

[2] U. Borisson, "Self-tuning regulators - industrial application and multivariable theory," tech. rep., Department of Automatic Control, Lund Institute of Technology, 1975. Report 7513.

[3] H. H. Rosenbrock, *State-space and Multivariable Theory.* New York: Nelson-Wiley, 1970.

[4] L. R. E. Shead, C. G. Anastassakis, and J. A. Rossiter, "Steady-state operability of multi-variable non-square systems: application to model predictive control (MPC) of the shell heavy oil fractionator (SHOF)," in *Proceedings of the 15th Mediterranean Conference on Control & Automation (MED'2007), Athens, Greece*, pp. 1–6, 2007. DOI: 10.1109/MED.2007.4433827.

[5] D. Horla, "On directional change and anti-windup compensation in multivariable control systems," *International Journal of Applied Mathematics and Computer Science*, vol. 19, no. 2, pp. 281 – 289, 2009. DOI: 10.2478/v10006-009-0024-4.

[6] D. Horla, "Directional change in multivariable lqg control with actuator failure," *Bulletin of the Polish Academy of Sciences: Technical Sciences*, vol. 65, no. No 4, pp. 419–428, 2017. DOI:10.1515/bpasts-2017-0047.

[7] E. D. Sontag, "On generalized inverses of polynomial and other matrices," *IEEE Transactions on Automatic Control*, vol. 25, no. 3, pp. 514–517, 1980. DOI: 10.1109/TAC.1980.1102377.

[8] A. Ben-Israel and T. N. E. Greville, *Generalized Inverses, Theory and Applications.* New York: Springer-Verlag, 2 ed., 2003.

[9] P. S. Stanimirović and M. D. Petković, "Computing generalized inverse of polynomial matrices by interpolation," *Applied Mathematics and Computation*, vol. 172, no. 1, pp. 508–523, 2006. DOI: 10.1016/j.amc.2005.02.031.

[10] F. M. Callier and F. Kraffer, "Proper feedback compensators for a strictly proper plant by polynomial equations," *International Journal of Applied Mathematics and Computer Science*, vol. 15, no. 4, pp. 493–507, 2005.

[11] B. A. Francis and P. P. Khargonekar, eds., *Robust Control Theory*, vol. 66 of *The IMA Volumes in Mathematics and its Applications.* New York: Springer-Verlag, 1994.

[12] A. Owczarkowski, D. Horla, and J. Zietkiewicz, "Introduction of feedback linearization to robust LQR and LQI control – analysis of results from an unmanned bicycle robot with reaction wheel," *Asian Journal of Control*, vol. 21, no. 2, pp. 1028–1040, 2019. DOI:10.1002/asjc.1773.

[13] L. Zhang, F. Zeng, W. Zhang, Y. Su, C. Zhou, S. Pan, and C. Ye, "Improvement of deadbeat control for pv converter," in *'in proceeding of the 2nd International Conference on Electrical, Control and Automation '*, 01 2017.

[14] H. K. Tam and J. Lam, "Robust deadbeat pole assignment with gain constraints: an LMI optimization approach," in *Optimal Control, Applications and Methods*, vol. 21, pp. 243–256, 2002. DOI:10.1002/oca.676.

[15] J.-F. Stumper., V. Hagenmeyer, S. Kuehl, and R. Kennel, "Deadbeat control for electrical drives: A robust and performant design based on differential flatness," in *IEEE Transactions on Power Electronics*, pp. 4585–4596, 2014. DOI:10.1109/TPEL.2014.2359971.

[16] N. Tsing, H. Tso, and J. Lam, "On the design of robust deadbeat regulators," in *IEEE Conference on Decision and Control*, 1996.

[17] A. Bartoszewicz and M. Maciejewski, "Sliding mode control of periodic review perishable inventories with multiple suppliers and transportation losses," *Bulletin of the Polish Academy of Sciences: Technical Sciences*, vol. 61, no. No 4, pp. 885–892, 2013.

[18] B. Zhang and J. Uhlmann, "A Generalized Matrix Inverse with Applications to Robotic Systems," *arXiv e-prints*, p. arXiv:1806.01776, May 2018.

[19] W. P. Hunek and K. J. Latawiec, "A study on new right/left inverses of nonsquare polynomial matrices," *International Journal of Applied Mathematics and Computer Science*, vol. 21, no. 2, pp. 331–348, 2011. DOI: 10.2478/v10006-011-0025-y.

[20] W. P. Hunek, "A new general class of MVC-related inverses of nonsquare polynomial matrices based on the Smith factorization," in *Proceedings of the 14th IEEE IFAC International Conference on Methods and Models in Automation and Robotics (MMAR'2009), Międzyzdroje, Poland*, 2009. Proceedings CD.

[21] N. P. Karampetakis and P. Tzekis, "On the computation of the generalized inverse of a polynomial matrix," *IMA Journal of Mathematical Control and Information*, vol. 18, no. 1, pp. 83–97, 2001. DOI: 10.1093/imamci/18.1.83.

[22] K. J. Latawiec, *The Power of Inverse Systems in Linear and Nonlinear Modeling and Control.* Opole, Poland: Opole University of Technology Press, 2004.

[23] W. P. Hunek and M. Krok, "Pole-free perfect control for nonsquare LTI discrete-time systems with two state variables," in *Proceedings of the 13th IEEE International Conference on Control and Automation (ICCA'2017), Ohrid, Macedonia*, pp. 329–334, 2017. DOI:10.1109/ICCA.2017.8003082.

[24] M. Krok and W. P. Hunek, "Pole-free vs. minimum-norm right inverse in design of minimum-energy perfect control for nonsquare state-space systems," in *Biomedical Engineering and Neuroscience; Proceedings of the 3rd International Scientific Conference on Brain-Computer Interfaces, BCI 2018, March 13-14, Opole, Poland*, Advances in Intelligent Systems and Computing, pp. 184–194, Springer, 2017. DOI:10.1007/978-3-319-75025-5_17.

[25] W. Hunek and M. Krok, "Parameter matrix σ-inverse in design of structurally stable pole-free perfect control for second-order state-space systems," in *Proceedings of the 24th International Conference on Automation and Computing (IEEE ICAC'18), Newcastle upon Tyne, England*, 2018. DOI: 10.23919/IConAC.2018.8748977.

[26] M. Krok and W. P. Hunek, "Pole-free perfect control: Theory vs. simulation examples," in *in Proceeding of the 23rd International Conference on Methods & Models in Automation & Robotics (MMAR'18), Międzyzdroje, Poland*, 2018. DOI: 10.1109/MMAR.2018.8486108.

[27] W. P. Hunek, "New interesting facts about minimum-energy perfect control for LTI nonsquare state-space systems," in *Proceedings of the 22th IEEE International Conference on Methods and Models in Automation and Robotics (MMAR'2017), Międzyzdroje, Poland*, pp. 274–278, 2017. DOI:10.1109/MMAR.2017.8046838.

[28] W. P. Hunek and M. Krok, "A study on a new criterion for minimum-energy perfect control in the state-space framework," *Proceedings of the Institution of Mechanical Engineers, Part I: Journal of Systems and Control Engineering*, vol. 233, no. 7, pp. 779–787, 2019. DOI:10.1177/0959651818823093.

[29] W. P. Hunek and M. Krok, "Towards a new minimum-energy criterion for nonsquare LTI state-space perfect control systems," in *Proceedings of the 5th International Conference on Control, Decision and Information Technologies, CoDIT 2018, Thessaloniki, Greece*, pp. 122–127, 2018. DOI:10.1109/CoDIT.2018.8394774.

[30] W. A. Wolowich, *Linear Multivariable Systems.* New York: Springer-Verlag, 1974.

[31] A. I. G. Vardulakis, N. P. Karampetakis, E. N. Antoniou, and E. Tictopoulou, “On the realization theory of polynomial matrices and the algebraic structure of pure generalized state space systems,” *International Journal of Applied Mathematics and Computer Science*, vol. 19, no. 1, pp. 77–88, 2009. DOI: 10.2478/v10006-009-0007-5.

[32] K. J. Latawiec, W. P. Hunek, and R. Stanisławski, “Selected topics in analysis, identification and control of LTI MIMO systems,” *Measurement Automation and Monitoring*, vol. 52, no. 10, pp. 25–28, 2006. special edition.

[33] M. Shad, A. Momeni, R. Errouissi, C. P. Diduch, M. E. Kaye, and L. Chang, “Identification and estimation for electric water heaters in direct load control programs,” *IEEE Transactions on Smart Grid*, vol. 8, pp. 947–955, March 2017.

[34] W. Favoreel, B. D. Moor, and P. V. Overschee, “Subspace state space system identification for industrial processes,” *Journal of Process Control*, vol. 10, no. 2, pp. 149 – 155, 2000.

[35] X.-M. Zhang, Q.-L. Han, A. Seuret, and F. Gouaisbaut, “An improved reciprocally convex inequality and an augmented Lyapunov–Krasovskii functional for stability of linear systems with time-varying delay,” *Automatica*, vol. 84, pp. 221 – 226, 2017.

[36] B. Wen, D. Boroyevich, R. Burgos, P. Mattavelli, and Z. Shen, “Inverse Nyquist stability criterion for grid-tied inverters,” *IEEE Transactions on Power Electronics*, vol. 32, pp. 1548–1556, Feb 2017.

[37] J. Klamka, “Controllability and minimum energy control problem of fractional discrete-time systems,” in *New Trends in Nanotechology and Fractional Calculus Applications* (D. Baleanu, Z. B. Guvenc, and J. A. T. Machado, eds.), pp. 503–509, Springer Netherlands, 2010. DOI:10.1007/978-90-481-3293-5_45.

[38] J. Klamka, *Controllability and Minimum Energy Control of Linear Finite Dimensional Systems*, pp. 13–25. Cham: Springer International Publishing, 2019. DOI:10.1007/978-3-319-92540-0_2.

[39] P. Lesniewski and A. Bartoszewicz, “Lq optimal control of periodic review perishable inventories with transportation losses,” in *Advances in Systems Science* (J. Swiątek, A. Grzech, P. Swiątek, and J. M. Tomczak, eds.), (Cham), pp. 45–55, Springer International Publishing, 2014. DOI: 10.1007/978-3-319-01857-7_5.

[40] M. Lass, T. D. Kühne, and C. Plessl, “Using approximate computing for the calculation of inverse matrix p-th roots,” *IEEE Embedded Systems Letters*, vol. 10, pp. 33–36, June 2018.

[41] T. Nguyen, “Inverse of a special matrix and application,” *CoRR*, vol. abs/1708.07795, 2017.

[42] W. P. Hunek and P. Majewski, “Perfect reconstruction of signal - a new polynomial matrix inverse approach,” *EURASIP Journal on Wireless Communications and Networking*, vol. 2018, no. 107, p. 8, 2018. DOI:10.1186/s13638-018-1122-5.

[43] W. P. Hunek, *Towards a General Theory of Control Zeros for LTI MIMO Systems*. Opole, Poland: Opole University of Technology Press, 2011.

[44] W. P. Hunek, “Pole-free vs. stable-pole designs of minimum variance control for nonsquare LTI MIMO systems,” *Bulletin of the Polish Academy of Sciences - Technical Sciences*, vol. 59, no. 2, pp. 201–211, 2011. DOI: 10.2478/v10175-011-0025-y.

[45] W. P. Hunek, K. J. Latawiec, R. Stanisławski, M. Łukaniszyn, and P. Dzierwa, “A new form of a σ-inverse for nonsquare polynomial matrices,” in *Proceedings of the 18th IEEE International Conference on Methods and Models in Automation and Robotics (MMAR'2013), Międzyzdroje, Poland*, pp. 282–286, 2013. DOI:10.1109/MMAR.2013.6669920.

[46] W. P. Hunek, “New SVD-based matrix H-inverse vs. minimum-energy perfect control design for state-space LTI MIMO systems,” in *Proceedings of the 20th IEEE International Conference on System Theory, Control and Computing (ICSTCC'16), Sinaia, Romania*, pp. 14–19, 2016. DOI: 10.1109/ICSTCC.2016.7790633.

[47] T. Kaczorek, “Cayley–Hamilton theorem for fractional linear systems,” in *Theory and Applications of Non-integer Order Systems* (A. Babiarz, A. Czornik, J. Klamka, and M. Niezabitowski, eds.), (Cham), pp. 45–55, Springer International Publishing, 2017.

[48] S. Paul, A. Halder, and A. K. Nath, “Deadbeat control of dynamics of inverted pendulum using signal correction technique,” *European Journal of Advances in Engineering and Technology*, vol. 4, no. 4, pp. 255–267, 2017.

[49] V. Kucera, “Deadbeat control, pole placement, and LQ regulation,” *Kybernetika*, vol. 35, no. 6, pp. 681–692, 1998.

[50] P. Van Dooren, “Deadbeat control: A special inverse eigenvalue problem,” *BIT Numerical Mathematics*, vol. 24, no. 4, pp. 681–699, 1984.

[51] D. Nesic, I. Mareels, G. Bastin, and R. Mahony, “Output dead beat control for a class of planar polynomial systems,” *SIAM journal on control and optimization*, vol. 36, no. 1, pp. 253–272, 1998.

[52] J. C. Engwerda, “Output deadbeat control using state feedback of discrete-time multivariable systems,” *International Journal of Systems Science*, vol. 26, no. 4, pp. 799–817, 1995. DOI: 10.1080/00207729508929069.

[53] M. Krok and W. P. Hunek, “Deadbeat vs. pole-free perfect control,” in *6th International Conference on Control, Decision and Information Technologies (CoDIT'19), Paris, France*, pp. 1338–1343, April 2019. DOI:10.1109/CoDIT.2019.8820611.

[54] S. Dadhich and W. Birk, “Analysis and control of extended quadruple tank process,” in *Proceedings of the 13th IEEE European Control Conference (ECC'2014)*, pp. 838–843, 2014. DOI:10.1109/ECC.2014.6862290.

[55] W. P. Hunek and T. Feliks, “A new geometric-oriented minimum-energy perfect control design in the imc-based state-space domain,” *IEEE Access*, vol. 8, pp. 41733–41739, 2020. DOI:10.1109/ACCESS.2020.2977278.

[56] W. P. Hunek, D. Paczko, T. Feliks, and M. Krok, “A norm-based approach to the minimum-energy multivariable perfect control design,” *2018 22nd International Conference on System Theory, Control and Computing (ICSTCC), Sinaia, Romania*, pp. 7–11, 2018. DOI: 10.1109/ICSTCC.2018.8540653.